Quantum inverse scattering method for the nonlinear Schrodinger model of fermions with attractive interaction

1989 J. Phys. A: Math. Gen. 22 4835

(http://iopscience.iop.org/0305-4470/22/22/014)

J. Phys. A: Math. Gen. 22 (1989) 4835–4851. Printed in the UK

Quantum inverse scattering method for the nonlinear Schrödinger model of fermions with attractive interaction

Haitao Fan†§, Fu-Cho Pu† and Bao-Heng Zhao‡

† Institute of Physics, Chinese Academy of Sciences, PO Box 603, Beijing, China
‡ Graduate School, Chinese Academy of Sciences, PO Box 3908, Beijing, China

Received 22 March 1989

Abstract. The quantum inverse scattering method is applied to the nonlinear Schrödinger model of fermions with attractive interaction. Both scattering and bound-state operators are constructed. Among state operators, there are a generator of infinitely many conserved quantities and the creators of the the eigenstates of these quantities. The commutation relations and the eigenvalues of physical interest are calculated. The quantum Gelfand–Levitan equations are established. The Fourier transform of the connected part of the 4-point Green function and the two body S matrix are calculated explicitly.

1. Introduction

In the development of the quantum inverse scattering method, the nonlinear Schrödinger model has played an important role. The direct problem of bosons of spin 0 with repulsive coupling was solved by Faddeev and Sklyanin (1978), Sklyanin (1979) and Thacker *et al* (1979). The correspond inverse problem was done by Creamer *et al* (1980). The direct generalisation of the work of Sklyanin (1979) and Pu and Zhao (1984) to the multicomponent nonlinear Schrödinger model of bosons or fermions with repulsive coupling was made by Pu *et al* (1987). Other generalisations with different emphasis were made by Kulish (1980, 1985). The system consisting of both bosons and fermions was studied by Fan *et al* (1988).

As is well known, there are bound states in the nonlinear Schrödinger model with attractive interaction. This makes the solving of the system much more difficult. Gockeler (1981a, b) introduced the bound-state operators in solving the system with attractive coupling. Although his method is rigorous and elegant, it is too complicated to apply to the multicomponent nonlinear Schrödinger model. In this paper, we generalise the simpler and rigorous approach of Pu and Zhao (1986) to solve the multicomponent nonlinear Schrödinger system with attractive coupling. In § 2, we define our model and introduce the auxiliary linear problem. We derive the commutation relations for some important operators by solving two sets of Yang–Baxter equations. Then, we construct the state operators and calculate commutation relations between them in § 3 and discuss these state operators in § 4. In §§ 5 and 6, we derive the Gelfand–Levitan equations of our model which are the central results of this paper. Finally, we apply these results to calculate the Fourier transform of the connected part of the 4-point Green function and the two body S matrix.

§ Present address: Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA.

2. The model and the auxiliary linear problem

Through this paper, we adopt the following convention: indices $a, b, c, d = 1, 2, \ldots, N$ and $l, m = 1, 2, \ldots, N+1$ where N is an even number; summation is taken for repeated indices.

The quantum nonlinear Schrödinger model of spin $(N-1)/2$ is defined by the Hamiltonian

$$H = \int_{-\infty}^{\infty} \left\{ \frac{\partial u^+(x)}{\partial x} \frac{\partial u(x)}{\partial x} + c u^+(x) u^+(x) u(x) u(x) \right\} \mathrm{d}x \tag{2.1}$$

where $c < 0$ is the coupling constant, $u(x) = (u_1(x), \ldots, u_N(x))$ are the Heisenberg field operators satisfying the equal time anticommutation relations:

$$\begin{aligned} &\{u_a^+(x), u_b(y)\} = \delta_{ab}\delta(x-y) \\ &\{u_a(x), u_b(x)\} = 0. \end{aligned} \tag{2.2}$$

The vacuum state $|0\rangle$ of the system is defined by $u(x)|0\rangle = 0$ and $\langle 0|0\rangle = 1$.

The Zakharov–Shabat auxiliary linear problem (Zakharov and Shabat 1971) for our system is

$$\begin{aligned} &\frac{\partial}{\partial x} T(x, y \mid \lambda) =: L(x, \lambda)\, T(x, y \mid \lambda) : \\ &T(y, y|\lambda) = 1 \end{aligned} \tag{2.3}$$

where ': ... :' denotes the normal order, λ is the spectral parameter and

$$L(x, \lambda) = \mathrm{i}\lambda J/2 - \sqrt{-c} E_{N+1,a} u_a^+(x) + \sqrt{-c} E_{a,N+1} u_a(x)$$

where $E_{ij}, i, j = 1, 2, \ldots, N+1$, and J are $(N+1) \times (N+1)$ matrices defined by

$$\begin{aligned} &(E_{ij})_{lm} = \delta_{il}\delta_{jm} \\ &J = \mathrm{diag}(1, \ldots, 1, -1). \end{aligned}$$

Starting with (2.3), we can obtain

$$\frac{\partial}{\partial x} (T(x, y \mid \lambda) \otimes_s T(x, y \mid \mu)) =: D(\lambda, \mu \mid x) \left[T(x, y \mid \lambda) \otimes_s T(x, y \mid \mu) \right] : \tag{2.4}$$

where

$$D(\lambda, \mu \mid x) = L(x, \lambda) \otimes_s I + I \otimes_s L(x, \mu) + c E_{N+1,a} \otimes E_{a,N+1}$$

where '$\otimes_s$' is the direct product of matrices in the graded sense, which is defined by

$$(U \otimes_s V)_{i,l:j,m} = (-1)^{p(l)[p(i)+p(j)]} U_{ij} V_{lm}$$

where U and V are $(N+1) \times (N+1)$ matrices and

$$p(l) = \begin{cases} 1 & \text{if } l = N+1 \\ 0 & \text{otherwise.} \end{cases}$$

After solving the Yang–Baxter equation

$$R(\lambda,\mu)D(\lambda,\mu \mid x) = D(\mu,\lambda \mid x)R(\lambda,\mu) \tag{2.5}$$

we get immediately the commutation relations

$$R(\lambda,\mu)[T(x,y \mid \lambda) \otimes_s T(x,y \mid \mu)] = [T(x,y \mid \mu) \otimes_s T(x,y \mid \lambda)]R(\lambda,\mu) \tag{2.6}$$

where

$$R(\lambda,\mu) = \frac{\mathrm{i}c}{\lambda-\mu+\mathrm{i}c} I_{N^2} + (-1)^{p(l)p(m)} \frac{\lambda-\mu}{\lambda-\mu+\mathrm{i}c} E_{l,m} \otimes E_{m,l}. \tag{2.7}$$

Now, we introduce some operators important to solving the model:

$$\varphi(x,\lambda) = \lim_{y\to\infty} E(-y,\lambda)T(y,x \mid \lambda) \tag{2.8}$$

$$\chi(x,\lambda) = \lim_{y\to-\infty} T(x,y \mid \lambda)E(y,\lambda) \tag{2.9}$$

where $E(y,\lambda) = \exp(\mathrm{i}\lambda y J/2)$.

The analyticity of $\varphi(x,\lambda)$ and $\chi(x,\lambda)$ are described by the following table:

Operators	Analytic region
$\varphi_{a,m}(x,\lambda)$, $\chi_{l,b}(x,\lambda)$	$\mathrm{Im}\,\lambda < 0$
$\varphi_{N+1,m}(x,\lambda)$, $\chi_{l,N+1}(x,\lambda)$	$\mathrm{Im}\,\lambda > 0$

After some manipulation on (2.6) similar to what was used by Thacker (1982) in which the existence of the limit of φ and χ as $y \to \pm\infty$ are taken into consideration, we get the following commutation relations for $\varphi(x,\lambda)$ and $\chi(x,\lambda)$:

$$R_+(\lambda,\mu)\left[\varphi(x,\lambda) \otimes_s \varphi(x,\mu)\right] = \left[\varphi(x,\mu) \otimes_s \varphi(x,\lambda)\right] \frac{R(\lambda,\mu)}{\lambda-\mu} \tag{2.10}$$

$$\frac{R(\lambda,\mu)}{\lambda-\mu}\left[\chi(x,\lambda) \otimes_s \chi(x,\mu)\right] = \left[\chi(x,\mu) \otimes_s \chi(x,\lambda)\right] R_-(\lambda,\mu) \tag{2.11}$$

where

$$\begin{aligned} R_+(\lambda,\mu) = {} & \frac{1}{\lambda-\mu} E_{a,b} \otimes \left[\frac{\mathrm{i}c}{\lambda-\mu+\mathrm{i}c}\delta_{ab}E_{c,c} + \frac{\lambda-\mu}{\lambda-\mu+\mathrm{i}c}E_{b,a} + \mathrm{i}\pi\delta_{ab}\delta(\lambda-\mu)E_{N+1,N+1}\right] \\ & + \frac{1}{\lambda-\mu+\mathrm{i}c}E_{N+1,a} \otimes E_{a,N+1} + \frac{\lambda-\mu-\mathrm{i}c}{(\lambda-\mu+\mathrm{i}0+)^2}E_{a,N+1} \otimes E_{N+1,a} \\ & - \mathrm{i}\pi\delta(\lambda-\mu)E_{N+1,N+1} \otimes E_{a,a}. \end{aligned}$$

$R_-(\lambda,\mu)$ can be obtained from the expression of $R_+(\lambda,\mu)$ by changing the sign of the δ-function term.

Similarly, we find

$$R_1(\lambda,\mu)[T^+(x,y\mid\lambda)\otimes_s T^{ST}(x,y\mid\mu)] = \left[T^{ST}(x,y\mid\mu)\otimes_s T^+(x,y\mid\lambda)\right] R(\lambda,\mu) \tag{2.12}$$

$$R_1(\lambda,\mu)\left[\varphi^+(x,\lambda)\otimes_s \varphi^{ST}(x,\lambda)\right] = \left[\varphi^{ST}(x,\mu)\otimes_s \varphi^+(x,\lambda)\right] R_{1+}(\lambda,\mu) \tag{2.13}$$

$$R_{1-}(\lambda,\mu)\left[\chi^+(x,\lambda)\otimes_s \chi^{ST}(x,\mu)\right] = \left[\chi^{ST}(x,\mu)\otimes_s \chi^+(x,\lambda)\right] R_1(\lambda,\mu) \tag{2.14}$$

where 'ST' is the supertranspose of matrices defined by

$$(T^{ST})_{lm} = (-1)^{p(l)[p(m)+1]} T_{ml}$$

and $T^+(x,y\mid\lambda) = [T(x,y\mid\bar{\lambda})]^+$:

$$R_1(\lambda,\mu) = E_{l,m}\otimes\left[(-1)^{p(l)+p(m)}\frac{ic}{\lambda-\mu}E_{l,m} + \frac{\lambda-\mu-iNc}{\lambda-\mu}(-1)^{p(l)p(m)}E_{m,l}\right]$$

$$\begin{aligned} R_{1-}(\lambda,\mu) = {} & E_{a,b}\otimes\left[\frac{ic}{\lambda-\mu}E_{a,b} + \frac{\lambda-\mu-iNc}{\lambda-\mu}E_{b,a}\right] \\ & + E_{N+1,a}\otimes\left[\frac{\lambda-\mu-iNc}{\lambda-\mu}E_{a,N+1} + \pi|c|\delta(\lambda-\mu)E_{N+1,a}\right] \\ & + E_{a,N+1}\otimes\left[\frac{\lambda-\mu-iNc}{\lambda-\mu}E_{N+1,a} - \pi|c|\delta(\lambda-\mu)E_{a,N+1}\right] \\ & - \frac{(\lambda-\mu-iNc)(\lambda-\mu-ic)}{(\lambda-\mu+i0+)^2}E_{N+1,N+1}\otimes E_{N+1,N+1}. \end{aligned}$$

The relation between $R_{1+}(\lambda,\mu)$ and $R_{1-}(\lambda,\mu)$ is the same as that between $R_-(\lambda,\mu)$ and $R_+(\lambda,\mu)$.

3. State operators

Just as in the nonlinear Schrödinger model of spin-0 particles, there are scattering states as well as bound states for our system when $c<0$. We start searching for the state operators corresponding to them with the following definitions:

$$\varphi(n,r\mid a_1,\ldots,a_{n-r}\mid\lambda,x) = \prod_{j=1}^{n-r}\varphi_{N+1,a_j}(x,\lambda_j)\prod_{j=n-r+1}^{n}\varphi_{N+1,N+1}(x,\lambda_j) \tag{3.1}$$

where $\lambda_j = \lambda + ic(n-1)/2 - ic(n-j)$. They are analytic with respect to λ with sufficient large $\mathrm{Im}\,\lambda$. An important property of $\varphi(\ldots)$ is

$$\varphi(n,r\mid Pa_1,Pa_2,\ldots,Pa_{n-r}\mid\lambda,x) = (-1)^{[P]}\varphi(n,r\mid a_1,\ldots,a_{n-r}\mid\lambda,x) \tag{3.2}$$

where P is a permutation of $a_1,\ldots,a_{n-r}$ and $(-1)^{[P]}$ is the sign of it. This antisymmetric property can be justified by

$$\varphi_{N+1,l}(x,\lambda+ic)\varphi_{N+1,m}(x,\lambda) = (-1)^{[p(l)+1][p(m)+1]}\varphi_{N+1,m}(x,\lambda+ic)\varphi_{N+1,l}(x,\lambda) \tag{3.3}$$

which is derived from (2.10). Now, (3.1) admits the following general Neumann expansion

$$\begin{aligned}
&\varphi(n,r \mid a_1,\ldots,a_{n-r} \mid \lambda,x)\\
&\quad= \sum_{k=0}^{\infty} \sum_{1\le b_1\le\ldots\le b_k\le N} \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \mathrm{d}x_1\ldots\mathrm{d}x_{n-r}\,\mathrm{d}y_1\ldots\mathrm{d}y_k\,\mathrm{d}z_1\ldots\mathrm{d}z_k\\
&\quad\times h_k\left(n,r\left|\begin{matrix}a_1\\x_1\end{matrix},\ldots,\begin{matrix}a_{n-r}\\x_{n-r}\end{matrix};\begin{matrix}b_1\\y_1\end{matrix},\ldots,\begin{matrix}b_k\\y_k\end{matrix}\right|\begin{matrix}b_k\\y_k\end{matrix},\ldots,\begin{matrix}b_1\\y_1\end{matrix}\left|\lambda,x\right.\right)\\
&\quad\times u^+_{a_1}(x_1)\ldots u^+_{a_{n-r}}(x_{n-r})u^+_{b_1}(y_1)\ldots u^+_{b_k}(x_k)u_{b_k}(z_k)\ldots u_{b_1}(z_1)
\end{aligned}\tag{3.4}$$

where $h_k(| \ldots | \ldots | \lambda,x)$ are antisymmetric with respect to the integral variables correspond to the indices in the same 'box'. By using the Gockeler (1981a) method, and the equation

$$\begin{aligned}
&\frac{\partial}{\partial x}\varphi(n,r \mid a_1,\ldots,a_{n-r} \mid \lambda,x)\\
&\quad= \tfrac{1}{2}\left[(2r-n)+\mathrm{i}c(n-r)r\right]\varphi(n,r \mid a_1,\ldots.a_{n-r} \mid \lambda,x)\\
&\quad+\sum_{n=1}^{n-r}(-1)^{j-1}\sqrt{-c}\,u^+_{a_j}(x)\,\varphi(n,r+1 \mid a_1,\ldots,\bar{a}_j,\ldots,a_{n-r} \mid \lambda,x)\\
&\quad-\sqrt{-c}\,r\,\varphi(n,r-1 \mid a_1,\ldots.a_{n-r},a \mid \lambda,x)u_a(x)
\end{aligned}\tag{3.5}$$

(where the bar on the index a_j means the absence of it), as well as the boundary condition

$$\lim_{x\to-\infty}\exp(\mathrm{i}\lambda nx/2)\,\varphi(n,r \mid a_1,\ldots,a_{n-r} \mid \lambda,x) = \delta_{n,r}\tag{3.6}$$

we can prove that $h_k(| \ldots | \ldots | \lambda,x)$ are analytic with respect to λ and when $c < 0$, $\operatorname{Im}\lambda > 0$

$$\begin{aligned}
&h_k(n,r \mid \ldots \mid \ldots \mid \lambda,x)\\
&\quad\le C_k(n,r)\prod_{j=1}^{n-r}\theta(x_j>x)\prod_{j=1}^{k}\theta(y_j>x)\theta(z_j>x)\exp\left\{\left[c(n-r)r/2-r\operatorname{Im}\lambda\right]x\right\}
\end{aligned}\tag{3.7}$$

where $k=0,1,\ldots$; $r=0,1,\ldots,n$ and $C_k(n,r)$ are constants. This implies the analyticity of $\varphi(n,r \mid a_1,\ldots,a_{n-r} \mid \lambda,x)$ on the upper half λ plane when $c<0$. Furthermore, we have

$$\lim_{x\to-\infty}\varphi(n,r \mid a_1,\ldots,a_{n-r} \mid \lambda,x) = 0 \qquad \text{for } r\neq 0,n.\tag{3.8}$$

We define the state operators $A_n(\lambda)$ and $B^+(\ldots)$ as the following:

$$A_n(\lambda) = \lim_{x\to-\infty}\exp(\mathrm{i}\lambda nx/2)\,\varphi(n,r \mid a_1,\ldots,a_{n-r} \mid \lambda,x)\tag{3.9}$$

$$B(\lambda \mid n \mid a_1,\ldots,a_n) = \lim_{x\to-\infty}\exp(-\mathrm{i}\lambda nx/2)\,\varphi(n,0 \mid a_1,\ldots,a_n \mid \lambda,x).\tag{3.10}$$

These state operators satisfy the following commutation relations which we will derive in appendix 1:

$$A_n(\lambda)A_m(\mu) = A_m(\mu)A_n(\lambda) \tag{3.11}$$

$$A_n(\lambda)B^+(\mu \mid n \mid a_1,\ldots,a_m) = \prod_{k=1}^{\bar{n}} \frac{\lambda-\mu-\mathrm{i}c(\mid n-m\mid +2k)/2}{\lambda-\mu-\mathrm{i}c(n+m-2k)/2+\mathrm{i}0+} B_m^+(\mu \mid n \mid a_1,\ldots,a_m)A_n(\lambda) \tag{3.12}$$

where $\bar{n} = \min(m,n)$;

$$A_n(\lambda)B(\mu \mid n \mid a_1,\ldots,a_m) = \prod_{k=1}^{\bar{n}} \frac{\lambda-\mu-\mathrm{i}c(n+m-2k)/2}{\lambda-\mu+\mathrm{i}c(\mid n-m\mid +2k)/2} B_m(\mu \mid n \mid a_1,\ldots,a_m)A_n(\lambda). \tag{3.13}$$

We list the commutation relations for $A_n(\lambda)$, $B(\ldots)$, $B^+(\ldots)$ etc in appendix 1.

4. The eigenstates of the system

Since $A_n(\lambda)$ and $A_m(\mu)$ commute, the expansion of $\ln A_1(\lambda)$ near $|\lambda| = \infty$

$$\ln A_1(\lambda) = \sum_{k=0}^{\infty} \frac{C_k}{(\mathrm{i}\lambda)^k} \tag{4.1}$$

will provide us with a family of commutative operators $\{C_k; k = 1,2,\ldots\}$. Further computation shows that the number of particles

$$N = -C_1/c$$

the total momentum

$$P = \mathrm{i}C_2/c - \mathrm{i}C_1/2$$

the total energy

$$H = C_3/c - C_2 + cC_1/6$$

and therefore $\{C_k; k = 1,2,\ldots,\}$ is a family of conserved quantities of our system which commute with each other.

Another kind of conserved quantities are generated by

$$T_{a,b}(\lambda) = \lim_{x\to-\infty} \exp(-\mathrm{i}\lambda x/2)\varphi_{a,b}(x,\lambda) \tag{4.2}$$

in the same way as (4.1). Among these quantities there are

$$N_a = \int_{-\infty}^{\infty} \mathrm{d}x\, u_a^+(x)u_a(x)$$

(a is fixed), namely the number of particles with a fixed value of the z component of spin.

Equation (3.10) indicates that

$$B^+(\lambda_1 \mid n_1 \mid a_1^{(1)},\ldots,a_{n_1}^{(1)})\ldots B^+(\lambda_k \mid n_k \mid a_1^{(k)},\ldots,a_{n_1}^{(k)}) \mid 0\rangle \tag{4.3}$$

where $k = 1,2,\ldots$ are the eigenstates of $A_n(\lambda)$ and thus common eigenstates of the infinite number of conserved quantities C_n, $n = 1,2,\ldots$. The eigenvalues of physical interest are given by the following:

$$[N, B^+(\lambda \mid n \mid a_1,\ldots,a_n)] = nB^+(\lambda \mid n \mid a_1,\ldots,a_n) \tag{4.4a}$$

$$[P, B^+(\lambda \mid n \mid a_1,\ldots,a_n)] = n\lambda B^+(\lambda \mid n \mid a_1,\ldots,a_n) \tag{4.4b}$$

$$[H, B^+(\lambda \mid n \mid a_1,\ldots,a_n)] = \left[n\lambda^2 - \frac{c^2}{12}(n^3 - n)\right] B^+(\lambda \mid n \mid a_1,\ldots,a_n) \tag{4.4c}$$

$$[H, A_n(\lambda)] = 0. \tag{4.4d}$$

This is equivalent to saying that the action of $B^+(\lambda \mid n \mid a_1,\ldots,a_n)$ on a state of our model is to add to the state an n-particle part (bound part if $n > 1$) with total momentum $n\lambda$ and total energy $[n\lambda^2 - \frac{1}{12}c^2(n^3 - n)]$, and the z component of the total spin $(\Sigma_{j=1}^n a_j) - n(N+1)/2$.

From (4.4c) and (4.4d), we derive the following time evolution relations for $B^+(\ldots)$ and $A_n(\lambda)$:

$$\begin{aligned}B^+(\mu \mid a_1,\ldots,a_n \mid t) &= \exp(\mathrm{i}Ht)B^+(\mu \mid a_1,\ldots,a_n)\exp(-\mathrm{i}Ht)\\ &= \exp\{\mathrm{i}t[n\mu^2 - c^2(n^3-n)/12]\}B^+(\mu \mid a_1,\ldots,a_n)\end{aligned} \tag{4.5}$$

$$A_n(\lambda, t) = A_n(\lambda). \tag{4.6}$$

Another fact implied from (3.2) is that there are only $2^N - 1$ different $B^+(\lambda \mid n \mid a_1,\ldots,a_n)$; N of them correspond to the scattering states of the system while the others correspond to the bound states.

In the following, we consider the states (4.3) as a complete base of the state space of our system and, therefore, we can prove equations and the analyticity of operators by evaluating the actions of these operators on the states (4.3)

5. Gelfand–Levitan equations (1)

We can prove the following by the definition (2.3) of $T(x,y \mid \lambda)$ that when $x \geq z \geq y$

$$T(x,z \mid \lambda)T(z,y \mid \lambda) = T(x,y \mid \lambda) \tag{5.1}$$

and

$$T^{-1}(x,y \mid \lambda - \mathrm{i}c) = \exp\left[c(x-y)/2\right] JT^+(x,y \mid \lambda)J. \tag{5.2}$$

Equation (5.1) tells us that $\varphi(x,\lambda)\chi(x,\lambda)$ is a constant with respect to x and we denote it by

$$T(\lambda) = \varphi(x,\lambda)\chi(x,\lambda). \tag{5.3}$$

In fact,

$$T_{N+1,N+1}(\lambda) = A_1(\lambda). \tag{5.4}$$

If we define

$$T^{(+)}(x,\lambda) = E(x,\lambda)\varphi(x,\lambda) \tag{5.5}$$

$$T^{(-)}(x,\lambda) = \chi(x,\lambda)E(-x,\lambda) \tag{5.6}$$

$$\left(\phi^{(l)}(x,\lambda)\right)^T = \left(T^{(+)}_{l,1}(x,\lambda),\dots,T^{(+)}_{l,N+1}(x,\lambda)\right) \tag{5.7}$$

$$\left(\psi^{(l)}(x,\lambda)\right)^T = \left(T^{(-)+}_{1,l}(x,\lambda),\dots,T^{(-)+}_{N+1,l}(x,\lambda)\right) \tag{5.8}$$

then (5.1) and (5.2) lead us to an Izergin–Korepin relation (cf Izergin and Korepin 1981) for the Jost functions $\phi^{(l)}(x,\lambda)$ and $\psi^{(l)}(x,\lambda)$:

$$\begin{aligned} G(x,\lambda) &= A_1^{-1}(\lambda-\mathrm{i}c)\phi^{(N+1)}(x,\lambda-\mathrm{i}c) \\ &= B^+(\lambda\mid 1\mid a)A_1^{-1}(\lambda)\psi^{(a)}(x,\lambda)\exp(-\mathrm{i}\lambda x)+\psi^{(N+1)}(x,\lambda) \end{aligned} \tag{5.9}$$

$$\begin{aligned} \frac{\partial}{\partial x}G(x,\lambda) =&: \begin{pmatrix} -\mathrm{i}\lambda I_N & \sqrt{-c}\,u^+(x) \\ \sqrt{-c}u(x) & 0 \end{pmatrix} G(x,\lambda) : \\ &+ ce_bF^{(a)}_{N+1}(x,\lambda)T^{(+)}_{N+1,b}(x,\lambda)A_1^{-1}(\lambda)T^{(-)+}_{N+1,a}(x,\lambda) \end{aligned} \tag{5.10}$$

where $e_b^T = (0,\dots,0,1,0,\dots,0)$ are $(N+1)$-dimensional vectors with bth component equal to 1, and

$$F^{(l)}_{N+1}(x,\lambda) = -B^+(\lambda\mid 1\mid a)A_1^{-1}(\lambda)T^{(-)}_{N+1,N+1}(x,\lambda)\exp(-\mathrm{i}\lambda x)+T^{(-)}_{N+1,a}(x,\lambda).$$

This hints to us to introduce

$$F^{(a)}(x,\lambda) = B^+(\lambda\mid 1\mid a)A_1^{-1}(\lambda)J\overline{\psi}^{(N+1)}(x,\lambda)\exp(-\mathrm{i}\lambda x)+\overline{\psi}^{(a)}(x,\lambda) \tag{5.11}$$

(where $\overline{\psi}^{(l)}(x,\lambda) = [\psi^{(l)+}(x,\lambda)]^T$) to form a complete group of differential equations along with (5.9):

$$\begin{aligned} \frac{\partial}{\partial x}F^{(a)}(x,\lambda) =&: \begin{pmatrix} 0 & \sqrt{-c}\,u(x) \\ -\sqrt{-c}\,u^+(x) & -\mathrm{i}\lambda \end{pmatrix} F^{(a)}(x,\lambda) : \\ &- ce_{N+1}F^{(a)}_{N+1}(x,\lambda)T^{(+)}_{N+1,b}(x,\lambda)A_1^{-1}(x,\lambda)T^{(-)}_{b,N+1}(x,\lambda). \end{aligned} \tag{5.12}$$

The boundary conditions of $G(x,\lambda)$ and $F^{(a)}(x,\lambda)$ at $x=\infty$ are

$$G(\infty,\lambda) = e_{N+1}\left[B^+(\lambda\mid 1\mid a)A_1^{-1}(\lambda)B(\lambda\mid 1\mid a)+A_1^+(\lambda)\right] \tag{5.13a}$$

$$\overline{F}^{(a)}(\infty,\lambda) = e_b\left[T^+_{b,a}(\lambda)+T^+_{b,N+1}(\lambda)A_1^{-1}(\lambda)B(\lambda\mid 1\mid a)\right]. \tag{5.13b}$$

When $\lambda \neq \mu$, we have

$$G(\infty,\lambda)B^+(\mu \mid n \mid a_1,\ldots,a_n) = \frac{\lambda-\mu-\mathrm{i}c(n+1)/2}{\lambda-\mu+\mathrm{i}c(n-1)/2}B^+(\mu \mid n \mid a_1,\ldots,a_n)G(\infty,\lambda)$$

$$\begin{aligned}\overline{F}^{(a)}(\infty,\lambda)B^+(\mu \mid n \mid a_1,\ldots,a_n) &= B^+(\mu \mid n \mid a_1,\ldots,a_n)\overline{F}^{(a)}_b(\infty,\lambda) \\ &\quad + \delta_{a,a_l}\frac{(-1)^j\mathrm{i}c}{\lambda-\mu+\mathrm{i}c(n-1)/2}B^+(\mu \mid n \mid d,a_1,\ldots,\overline{a}_l,\ldots,a_n)\overline{F}^{(d)}_b(\infty,\lambda)\end{aligned}$$

$$A_1^{-1}(\lambda)B^+(\mu \mid n \mid a_1,\ldots,a_n) = \frac{\lambda-\mu-\mathrm{i}c(n-1)/2}{\lambda-\mu+\mathrm{i}c(n+1)/2}B^+(\mu \mid n \mid a_1,\ldots,a_n)A_1^{-1}(\lambda)$$

We conclude from these that $G(\infty,\lambda)$, $F^{(l)}(\infty,\lambda)$ and $A^{-1}(\lambda)$ are piecewise analytic on the upper half λ plane with discontinuities across $\mathrm{Im}\,\lambda = -cn/2$ where $n = 1,2,\ldots,N-1$ for $G(\infty,\lambda)$ and $F^{(a)}(\infty,\lambda)$ while $n = 2,3,\ldots,N+1$ for $A^{-1}(\lambda)$. Thus, we can expand $G(x,\lambda)$ and $F^{(a)}(x,\lambda)$ as we did for $\varphi(\ldots)$ in § 3 and analyse the analyticity of $G(x,\lambda)$ and $F^{(a)}(x,\lambda)$ with the help of the differential equations (5.10) and (5.12). We can find that $G(x,\lambda)$ and $F^{(a)}(x,\lambda)$ are analytic except when $\mathrm{Im}\,\lambda = -cn/2$, $n = 1,2,\ldots,N+1$. On the other hand, the equation

$$G(x,\lambda) = A^{-1}(\lambda-\mathrm{i}c)\phi^{(N+1)}(x,\lambda-\mathrm{i}c)$$

enables us to rule out two of them: $\mathrm{Im}\,\lambda = -cN/2$, $-c(N+1)/2$ for $G(x,\lambda)$.

Now, using the analyticity of $G(x,\lambda)$ and $F^{(a)}(x,\lambda)$ as well as their asymptotic behaviour

$$G(x,\lambda) \sim e_{N+1} + \mathrm{O}(\frac{1}{\lambda})$$

$$F^{(a)}(x,\lambda) \sim e_a + \mathrm{O}(\frac{1}{\lambda})$$

we can study some contour integrals and write out a series of integral equations

$$\begin{aligned}\psi^{(N+1)}(x,\lambda) &= e_{N+1} - \frac{1}{2\pi i}\int_{-\infty}^{\infty} d\mu \frac{B^+(\mu \mid 1 \mid a)A_1^{-1}(\mu)\psi^{(a)}(x,\mu)}{\lambda-\mu-\mathrm{i}0+}\exp(-\mathrm{i}\mu x) \\ &\quad - \frac{1}{2\pi\mathrm{i}}\sum_{n=1}^{N-1}\int_{-\infty}^{\infty}\mathrm{d}\mu\frac{\mathrm{disc}\,G(x,\mu-\mathrm{i}cn/2)}{\lambda-\mu+\mathrm{i}cn/2}\end{aligned} \tag{5.14a}$$

$$\begin{aligned}\psi^{(a)}(x,\lambda) &= e_a + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}\mu\frac{J\psi^{(N+1)}(x,\mu)A_1^{-1}(\mu)B(\mu \mid 1 \mid a)}{\lambda-\mu+\mathrm{i}0+}\exp(\mathrm{i}\mu x) \\ &\quad - \frac{1}{2\pi\mathrm{i}}\sum_{n=1}^{N+1}\int_{-\infty}^{\infty}\mathrm{d}\mu\frac{\mathrm{disc}\,\overline{F}^{(a)}(x,\mu+inc/2)}{\lambda-\mu-\mathrm{i}cn/2}\end{aligned} \tag{5.14b}$$

where $\mathrm{disc}\,G(x,\mu) = G(x,\mu+\mathrm{i}0+) - G(x,\mu-\mathrm{i}0+)$.

6. Gelfand–Levitan equations (2)

6.1. The expression of disc (x, λ)

In the previous section, we obtained a series of integral equations of our system. But, we still do not know much about disc $G(\ldots)$ and disc $F^{(a)}(\ldots)$. We are going to find a expression for them in this section.

First of all, we can prove that disc $G(x, \lambda)$ can be expressed in the form

$$\operatorname{disc} G^T(x,\lambda) = \big(H_1(x,\lambda),\ldots,H_N(x,\lambda)\big)\begin{pmatrix}\psi^{(1)T}(x,\lambda)\\ \vdots \\ \psi^{(1)T}(x,\lambda)\end{pmatrix}\exp(-\mathrm{i}\lambda x) \tag{6.1.1}$$

by using disc $A^{-1}(\lambda)\times A(\lambda)=0$ and the following consequence of (5.2):

$$\begin{pmatrix} T_{11}^{(-)+}(x,\lambda) & \ldots & T_{N1}^{(-)+}(x,\lambda)\\ \vdots & \ldots & \vdots \\ T_{1N}^{(-)+}(x,\lambda) & \ldots & T_{NN}^{(-)+}(x,\lambda)\end{pmatrix}^{-1}\times\begin{pmatrix}T_{N+1,1}^{(-)+}(x,\lambda)\\ \vdots \\ T_{N+1,N+1}^{(-)+}(x,\lambda)\end{pmatrix}$$
$$= -\begin{pmatrix}T_{1,N+1}^{(-)}(x,\lambda-\mathrm{i}c)\\ \vdots \\ T_{N,N+1}^{(-)}(x,\lambda-\mathrm{i}c)\end{pmatrix}[T_{N,N+1}^{(-)}(x,\lambda-\mathrm{i}c)]^{-1}. \tag{6.1.2}$$

Taking $\partial/\partial x$ on both sides

$$\operatorname{disc} A_1^{-1}(\lambda-\mathrm{i}c)\phi^{(N+1)T}(x,\lambda-\mathrm{i}c) = \operatorname{disc} G^T(x,\lambda)$$
$$= (H_1(x,\lambda),\ldots,H_N(x,\lambda))\begin{pmatrix}\phi^{(1)T}(x,\lambda)\\ \vdots \\ \psi^{(N)T}(x,\lambda)\end{pmatrix}\exp(-\mathrm{i}\lambda x) \tag{6.1.3}$$

and using

$$u_b^+(x)\begin{pmatrix}\psi^{(1)T}x,\lambda)\\ \vdots \\ \psi^{(N)T}(x,\lambda)\end{pmatrix}J = \begin{pmatrix}\psi^{(1)T}(x,\lambda)\\ \vdots \\ \psi^{(N)T}(x,\lambda)\end{pmatrix}\left[u_b^+(x)-\tfrac{1}{2}\sqrt{-c}\,E_{b,N+1}\right] \tag{6.1.4}$$

$$u_b^+(x)\phi^{(N+1)+}(x,\lambda)J = -\phi^{(N+1)+}(x,\lambda)\left[u_b^+(x)-\tfrac{1}{2}\sqrt{-c}\,E_{b,N+1}\right] \tag{6.1.5}$$

and (5.1) and (5.3), we find that for arbitrary y

$$\left(\frac{\partial}{\partial x}H_1(x,\lambda),\ldots,\frac{\partial}{\partial x}H_N(x,\lambda)\right)\begin{pmatrix}\psi^{(1)T}(y,\lambda)\\ \vdots \\ \psi^{(N)T}(x,\lambda)\end{pmatrix}=0 \tag{6.1.6}$$

This means that $H_a(x,\lambda)$ are constants with respect to x and therefore

$$\operatorname{disc} G^T(x,\lambda) = \big(H_1(\lambda),\ldots,H_N(\lambda)\big)\begin{pmatrix}\psi^{(1)T}(x,\lambda)\\ \vdots \\ \psi^{(N)T}(x,\lambda)\end{pmatrix}\exp(-\mathrm{i}\lambda x). \tag{6.1.7}$$

On the other hand, it can be shown by calculating the eigenvalues of $G(x, \lambda)$ corresponding to (4.3) that

$$\operatorname{disc} G(\infty, \lambda - \mathrm{i}c(n-1)/2) = e_{N+1} B^{+}(\lambda \mid n \mid a_1, \ldots, a_n) A_n^{-1}(\lambda) \left(A_{n-1}^{+}(\lambda - \mathrm{i}c/2)\right)^{-1} B(\lambda \mid n \mid a_1, \ldots, a_n). \tag{6.1.8}$$

This equation and the definition of $B^{+}(\ldots)$ lead us to the conjecture that

$$\begin{aligned}\operatorname{disc} G(x, \lambda - \mathrm{i}c(n-1)/2) &= B^{+}(\lambda \mid n \mid a, a_1, \ldots, a_{n-1}) \times A_n^{-1}(\lambda) \left[A_{n-1}^{+}(\lambda + \mathrm{i}c/2)\right]^{-1} \\ &\quad \times B(\lambda + \mathrm{i}c/2 \mid n-1 \mid a_1, \ldots, a_{n-1}) \psi^{(a)}(x, \lambda - \mathrm{i}c(n-1)/2) \\ &\quad \times \exp\{-\mathrm{i}x[\lambda - \mathrm{i}c(n-1)/2]\}\end{aligned} \tag{6.1.9}$$

that is

$$H_a(\lambda - \mathrm{i}c(n-1)/2) = R^{+}(\lambda \mid n \mid a, a_1, \ldots, a_{n-1}) R(\lambda + \mathrm{i}c/2 \mid n-1 \mid a_1, \ldots, a_{n-1}) \tag{6.1.10}$$

where

$$R^{+}(\lambda \mid n \mid a_1, \ldots, a_n) = B^{+}(\lambda \mid n \mid a_1, \ldots, a_n) A_n^{-1}(\lambda).$$

In appendix 2, we will prove this conjecture and justify the expression

$$\sum_P (-1)^{[P]} B(\lambda + \mathrm{i}c/2 \mid n-1 \mid Pa_1, \ldots, Pa_{n-1}) \psi^{(Pa_n)}(x, \lambda - \mathrm{i}c(n-1)/2)$$

as the analytic continuation of

$$\sum_P (-1)^{[P]} B(\lambda \mid n-1 \mid Pa_1, \ldots, Pa_{n-1}) \psi^{(Pa_n)}(x, \lambda - \mathrm{i}cn/2)$$

where P are the permutations of $a_1, \ldots, a_n$.

6.2. *The Gelfand–Levitan equations*

By computing the following:

$$u_b^{+}(x) G(x, \lambda) + J G(x, \lambda) u_b^{+}(x)$$

we find the relation between $H_a(\lambda)$ and $\operatorname{disc} F_{(b)}(x, \lambda)$:

$$\begin{aligned}&\left(\{u_b(x), H_1(\lambda)\}, \ldots, \{u_b(x), H_1(\lambda)\}\right) \begin{pmatrix} \psi^{(1)T}(y, \lambda) \\ \vdots \\ \psi^{(N)T}(y, \lambda) \end{pmatrix} \exp(-\mathrm{i}\lambda x) \\ &\quad = \sqrt{-c}\, \operatorname{disc}\left[\left(F_{N+1}^{(1)}(x, \lambda), \ldots, F^{(N)}(x, \lambda)\right) T_{N+1,b}^{(+)}(x, \lambda) A_1^{-1}(\lambda)\right] \begin{pmatrix} \psi^{(1)T}(y, \lambda) \\ \vdots \\ \psi^{(N)T}(y, \lambda) \end{pmatrix}.\end{aligned} \tag{6.2.1}$$

This means that if the action of one of the $\{u_b(x), H_a(\lambda)\}\exp(-i\lambda x/2)$ and

$$\mathrm{disc}\left[F^{(a)}(x,\lambda)T^{(+)}_{N+1,b}(x,\lambda)A_1^{-1}(\lambda)\right]$$

on the states

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\mathrm{d}x_1\ldots\mathrm{d}x_m\prod_{j=1}^{m}\theta(y<x_j)\,g(y;x_1,\ldots,x_m)\,u^+_{a_1}(x)\ldots u^+_{a_m}(x)\mid 0\rangle \tag{6.2.2}$$

is well defined, then so is the other and they are equal:

$$\{u_b(x), H_a(\lambda)\}\exp(-i\lambda x/2) = \mathrm{disc}\left[F^{(a)}(x,\lambda)T^{(+)}_{N+1,b}(x,\lambda)A_1^{-1}(\lambda)\right]. \tag{6.2.3}$$

This conclusion leads us to the discovery that if

$$\Delta^{(a)}(x,\lambda) = H_a(\lambda)J\overline{\psi}^{(N+1)}(x,\lambda) \tag{6.2.4}$$

is well defined, then

$$\begin{aligned}&\frac{\partial}{\partial x}\left[\Delta^{(a)}(x,\lambda)-\mathrm{disc}\,F^{(a)}(x,\lambda)\right]\\ &\quad = :\begin{pmatrix}0 & \sqrt{-c}\,u(x)\\ -\sqrt{-c}\,u^+(x) & -i\lambda\end{pmatrix}\left[\Delta^{(a)}(x,\lambda)-\mathrm{disc}\,F^{(a)}(x,\lambda)\right]:.\end{aligned} \tag{6.2.5}$$

On the other hand, after evaluating the action of $\Delta^{(a)}(\infty,\lambda)+\mathrm{disc}\,\bar{F}^{(a)}(\infty,\lambda)$ on states (4.3), we find that it is zero. According to the uniqueness of the initial value problem of (6.2.5), it follows that

$$\begin{aligned}&\mathrm{disc}\,\bar{F}^{(a)}(x,\lambda+ic(n-1)/2)\\ &\quad = J\psi^{(N+1)}(x,\lambda+ic(n-1)/2)R^+(\lambda-ic/2\mid n-1\mid a_1,\ldots,a_{n-1})\\ &\qquad\times R(\lambda\mid n\mid a,a_1,\ldots,a_{n-1}).\end{aligned} \tag{6.2.6}$$

We accomplish the above derivation by proving, in appendix 3, that $\Delta^{(a)}(x,\lambda)$ and $\Delta^{(a)}(\infty,\lambda)$ are well defined. Now, we can replace the disc $G(x,\lambda)$ and disc $F(x,\lambda)$ in (5.14) by (6.2.6) and (6.1.9) and obtain the Gelfand–Levitan equations of our system:

$$\begin{aligned}\psi^{(N+1)}(x,\lambda) &= e_{N+1}-\frac{1}{2\pi i}\sum_{n=1}^{N}\int_{-\infty}^{\infty}\mathrm{d}\mu\,\frac{\exp\{ix[\mu-ic(n-1)/2]\}}{\lambda-\mu+ic(n-1)/2-i0+}\\ &\quad\times R^+(\mu\mid n\mid a,a_1,\ldots,a_{n-1})R(\mu+ic/2\mid n-1\mid a_1,\ldots,a_{n-1})\\ &\quad\times\psi^{(a)}(x,\lambda-ic(n-1)/2)\end{aligned} \tag{6.2.7a}$$

$$\begin{aligned}\psi^{(a)}(x,\lambda) &= e_a+\frac{1}{2\pi i}\sum_{n=1}^{N}\int_{-\infty}^{\infty}\mathrm{d}\mu\,\frac{\exp\{ix[\mu+ic(n-1)/2]\}}{\lambda-\mu-ic(n-1)/2+i0+}\,J\psi^{(N+1)}(x,\lambda+ic(n-1)/2)\\ &\quad\times R^+(\mu-ic/2\mid n-1\mid a_1,\ldots,a_{n-1})R(\mu\mid n\mid a,a_1,\ldots,a_{n-1}).\end{aligned} \tag{6.2.7b}$$

Expanding the $(N+1)$th component of $\psi^{(a)}(x,\lambda)$ near $\lambda=\infty$, we find that

$$\psi^{(a)}_{N+1}(x,\lambda)=\frac{\mathrm{i}\sqrt{-c}}{\lambda}u_b(x)+\mathrm{O}(\frac{1}{\lambda^2}). \tag{6.2.8}$$

Applying (6.2.8) and (4.5) to (6.2.7), we find the time evolution for the Heisenberg field operators:

$$\begin{aligned}u_a(x,t)=&\frac{1}{2\pi\sqrt{-c}}\int_{-\infty}^{\infty}\mathrm{d}\mu R(\mu\mid 1\mid a)\exp(\mathrm{i}\mu x-\mathrm{i}\mu^2 t)\\&-\frac{1}{(2\pi)^3\sqrt{-c}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\mu_1\,\mathrm{d}\mu_2\,\mathrm{d}\mu_3\\&\times\Bigg[\exp[\mathrm{i}x(\mu_1+\mu_3-\mu_2)+\mathrm{i}t(\mu_1^2-\mu_1^2-\mu_3^2)]\\&\times\frac{R^+(\mu_2\mid 1\mid b)R(\mu_3\mid 1\mid b)R(\mu_1\mid 1\mid a)}{(\mu_2-\mu_1+\mathrm{i}0+)(\mu_3-\mu_2-\mathrm{i}0+)}\Bigg]-\frac{\sqrt{-c}}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}\mu_1\,\mathrm{d}\mu_2\\&\times\Bigg[\exp[\mathrm{i}x(2\mu_1-\mu_2)+\mathrm{i}t(\mu_2^2-2\mu_1^2+3c^2)]\,\frac{R^+(\mu_2\mid 1\mid b)R(\mu_1\mid 2\mid b,a)}{(\mu_2-\mu_1)^2+c^2/4}\Bigg]\\&+\ldots.\end{aligned} \tag{6.2.9}$$

By using the above results, we can calculate explicitly the Fourier transformation of the connected part of the 4-point Green function and the two body S matrix. They are

$$G^{4c}=\frac{-8\pi^2\mathrm{i}c\left[1+(-1)^J\right]\delta(k_1+k_2-k_1'-k_2')\delta(\omega_1+\omega_2-\omega_1'-\omega_2')}{\Pi_{j=1}^2\left[(k_j^2-\omega_j-\mathrm{i}0+)(k_j'^2-\omega_j'-\mathrm{i}0+)\right](1+\mathrm{i}c\sigma^{-1})}$$

and

$$S=(-1)^J\frac{\mathrm{i}c}{\lambda-\mu+\mathrm{i}c}-\frac{\lambda-\mu}{\lambda-\mu+\mathrm{i}c}$$

respectively, where J is the total spin of the two particle states and

$$\sigma=\left[2(\omega_1+\omega_2)-(k_1+k_2)^2\right]^{1/2}.$$

Appendix 1. The commutation relations of state operators

A natural way to get the commutation relations for state operators is to calculate the corresponding relations for $\varphi(\ldots)$. For example, we study the commutation relation between $\varphi(n,n\mid\lambda,x)$ and $\varphi(m,0\mid a_1,\ldots,a_m\mid\mu,x)$ in the attempt to find the one between $A_n(\lambda)$ and $B(\mu\mid m\mid a_1,\ldots,a_m)$.

Lemma.

$$\varphi(n, r \mid a_1, \ldots, a_{n-r} \mid \lambda, x)\varphi(m, s \mid b_1, \ldots, b_{m-s} \mid \mu, x)$$

can be expressed by the linear combination of

$$\varphi(m, s' \mid b'_1, \ldots, b'_{m-s'} \mid \mu, x)\varphi(n, r' \mid a'_1, \ldots, a'_{n-r'} \mid \lambda, x)$$

where $b'_1, \ldots, b'_{m-s'}, a'_1, \ldots, a'_{n-r'}$ are the rearrangements of $a_1, \ldots, a_{n-r}, b_1, \ldots, b_{m-s}$.

Therefore,

$$\begin{aligned}\varphi(m, 0 \mid a_1, \ldots, a_m \mid \mu, x)&\varphi(n, n \mid \lambda, x)\\ &= c_1\varphi(n, n \mid \lambda, x)\varphi(m, 0 \mid a_1, \ldots, a_m \mid \mu, x)\\ &\quad + c_2\delta_{mn}\varphi(n, 0 \mid a_1, \ldots, a_n \mid \mu, x)\varphi(n, n \mid \lambda, x) + \ldots . \end{aligned} \tag{A1.1}$$

According to (3.6), all the other terms vanish as $x \to -\infty$. Thus, the only thing we have to do now is to calculate the coefficients c_1, c_2 in (A1.1). We accomplish this by using

$$\begin{aligned}\varphi_{N+1,l}(x, \lambda)&\varphi_{N+1,m}(x, \mu)\\ &= (-1)^{[p(l)+1][p(m)+1]}\frac{\lambda - \mu}{\lambda - \mu - \mathrm{i}c}\varphi_{N+1,m}(x, \mu)\varphi_{N+1,l}(x, \lambda)\\ &\quad - \frac{\mathrm{i}c}{\lambda - \mu - \mathrm{i}c}\varphi_{N+1,l}(x, \lambda)\varphi_{N+1,m}(x, \mu). \end{aligned} \tag{A1.2}$$

We have

$$\begin{aligned}\varphi(n, n \mid \lambda, x)&\varphi(m, 0 \mid a_1, \ldots, a_m \mid \mu, x)\\ &= \prod_{k=1}^{\bar{n}}\frac{\lambda - \mu + \mathrm{i}c(\mid n - m \mid +2k)/2}{\lambda - \mu - \mathrm{i}c(m + n - 2k)/2}\varphi(m, 0 \mid a_1, \ldots, a_m \mid \mu, x)\varphi(n, n \mid \lambda, x)\\ &\quad + \frac{(\mathrm{i}c)^n n!\delta_{mn}}{\Pi_{k=1}^{n}[\lambda - \mu - \mathrm{i}c(n - k)]}\varphi(n, 0 \mid a_1, \ldots, a_n \mid \mu, x)\varphi(n, n \mid \lambda, x) + \ldots . \end{aligned} \tag{A1.3}$$

By letting $x \to -\infty$ in (A1.3), we get (3.13).

Similarly, we can obtain other commutation relations among $A_n(\lambda)$, $B^+(\ldots)$ and $B(\ldots)$. We list some of them in the following:

$$\begin{aligned}B^+(\lambda \mid n \mid a_1, \ldots, a_{n-r}, c_1, \ldots, c_r)&B^+(\mu \mid m \mid b_1, \ldots, b_{m-r}, c_1, \ldots, c_r)\\ &= \sum_{l=0}^{\bar{n}}\sum_{(1\le i_1<\ldots<i_l\le n-r)}\sum_{(1\le j_1<\ldots<j_l\le m-r)}(-1)^{mn+l(n-r)+\sum_{k=1}^{l}(i_k+j_k)}(\mathrm{i}c)^l l!\\ &\quad \times\frac{\Pi_{k=1}^{\bar{n}-r-l}[\lambda - \mu + \mathrm{i}c(\mid n - m \mid +2k + 2)/2]}{\Pi_{k=1}^{\bar{n}}[\lambda - \mu - \mathrm{i}c(m + n + 2 - 2k)]}\prod_{k=\bar{n}-r+1}^{\bar{n}}[\lambda - \mu + \mathrm{i}c(\mid n - m \mid +2k)/2]\\ &\quad \times B^+(\mu \mid m \mid a_{i_1}, \ldots, a_{i_l}, b_1, \ldots, \bar{b}_{j_1}, \ldots, \bar{b}_{j_l}, \ldots, b_{m-r}, c_1, \ldots, c_r)\\ &\quad \times B^+(\lambda \mid n \mid a_1, \ldots, \bar{a}_{i_1}, \ldots, \bar{a}_{i_l}, \ldots, a_{n-r}, b_{j_1}, \ldots, \bar{b}_{j_l}, c_1, \ldots, c_r) \end{aligned} \tag{A1.4}$$

where the sets $\{a_1,\ldots,a_{n-r}\}$ and $\{b_1,\ldots,b_{m_r}\}$ have no common element and $c_1,\ldots,c_r$ are fixed. The bar on the indices means the absence of them:

$$
\begin{aligned}
B(\lambda \mid n \mid a_1,\ldots,a_{n-r},c_1,\ldots,c_r)&B^+(\mu \mid m \mid b_1,\ldots,b_{m-r},c_1,\ldots,c_r) \\
&= \sum_{l=0}^{r} \sum_{(1\le i_1<,\ldots,i_l\le r)} \sum_{(d_{i_j}\neq c_1,\ldots,c_r)} (-1)^{mn-l}(\mathrm{i}c)^l \\
&\quad\times \frac{\Pi_{k=1}^{\bar{n}-r-l}[\lambda-\mu-\mathrm{i}c(\mid n-m\mid+2k-2)/2]}{\Pi_{k=1}^{\bar{n}}[\lambda-\mu+\mathrm{i}c(m+n+2-2k)]} \\
&\quad\times \prod_{k=\bar{n}-r+l+1}^{\bar{n}} [\lambda-\mu-\mathrm{i}c(\mid n-m\mid+2k)/2]\frac{(\lambda-\mu)^2+c^2(m+n)^2/4}{(\lambda-\mu)^2+c^2(n-m)^2} \\
&\quad\times B^+(\mu \mid m \mid b_1,\ldots,b_{m-r},d_{i_1},\ldots,d_{i_l},c_1,\ldots,\mathrm{i}\bar{c}_{i_1},\ldots,\bar{c}_{i_l},\ldots,c_r) \\
&\quad\times B(\lambda \mid n \mid a_1,\ldots,a_{n-r},d_{i_1},\ldots,d_{i_l},c_1,\ldots,\mathrm{i}\bar{c}_{i_1},\ldots,\bar{c}_{i_l},\ldots,c_r) \\
&\quad+ \frac{2\pi\mid c\mid}{(n-1)!}\delta_{mn}\delta_{nr}\delta(\lambda-\mu)A_n^+(\lambda)A_n(\lambda).
\end{aligned} \tag{A1.5}
$$

Appendix 2. The proof of (6.1.9)

Definition.

$$
\begin{aligned}
\chi(n,r \mid b_1,\ldots,b_{n-r} \mid a_1,\ldots,a_n \mid \lambda,x) \\
= \sum_P (-1)^{[P]} \prod_{j=1}^{r} \chi_{N+1,Pa_j}(x,\lambda_{n-j+1}) \prod_{j=r+1}^{n} \chi_{b_{j-r},Pa_j}(x,\lambda_{n-j+1})
\end{aligned} \tag{A2.1}
$$

where $\lambda_j = \lambda + \mathrm{i}c(n-1)/2 - \mathrm{i}c(n-j)$.

From (2.11), we can derive that

$$
\chi_{la}(x,\lambda+\mathrm{i}c)\chi_{mb}(x,\lambda)-\chi_{lb}(x,\lambda+\mathrm{i}c)\chi_{ma}(x,\lambda) = \chi_{mb}(x,\lambda+\mathrm{i}c)\chi_{la}(x,\lambda)-\chi_{ma}(x,\lambda+\mathrm{i}c)\chi_{lb}(x,\lambda). \tag{A2.2}
$$

Now, we can prove the analyticity of $\chi(n,r \mid \ldots \mid \lambda,x)$ on the lower half λ plane by using the same method we used to prove the analyticity of $\varphi(n,r \mid \ldots \mid \lambda,x)$.

Operators $\chi(n,r \mid \ldots \mid \lambda,x)$ along with $\varphi(n,r \mid \ldots \mid \lambda,x)$ gives us an alternative expression of $B^+(\ldots)$:

$$
\begin{aligned}
B^+(\lambda \mid n \mid a_1,\ldots,a_n) \\
= \frac{1}{n!}\sum_{r=0}^{n}(-1)^{(n+1)r}\binom{n}{r}\varphi(n,r \mid b_1,\ldots,b_{n-r} \mid \lambda,x) \\
\times \chi(n,r \mid b_1,\ldots,b_{n-r} \mid a_1,\ldots,a_n \mid \lambda,x).
\end{aligned} \tag{A2.3}
$$

Thus we can make the right-hand side of (6.1.9) well defined by introducing the expression

$$\sum_{j=1}^{n}(-1)^{j-1}B(\lambda+ic/2 \mid n-1 \mid a_1,\ldots\bar{a}_j,\ldots,a_n)\, T^{(-)+}_{m,a_j}(x,\lambda-ic(n-1)/2)$$

$$= \sum_{r=0}^{n-1}\frac{(-1)^{(1-\delta_{m,N+1})r}}{(n-1)!}\binom{n-1}{r}\exp\{\tfrac{1}{2}i[\lambda-ic(n-1)/2]x\}$$
$$\times\, \varphi^{+}(n-1,r \mid b_1,\ldots,b_{n-r-1} \mid \lambda+ic/2,x)$$
$$\times\, \chi^{+}(n,r+\delta_{m,N+1} \mid (1-\delta_{m,N+1})m,b_1,\ldots,b_{n-r-1} \mid a_1,\ldots a_n \mid \lambda,x). \qquad \text{(A2.4)}$$

Under this definition, the right-hand side of (6.1.9) has the same boundary value as that of $\operatorname{disc} G(x,\lambda)$ at $x=\infty$. This implies that the action of

$$\operatorname{disc} G(x,\lambda-ic(n-1)/2)-H_a(\lambda-ic(n-1)/2)\psi^{(a)}(x,\lambda-ic(n-1)/2)\exp\{-ix[\lambda-ic(n-1)/2]\}$$

on the states

$$\mid f(m,y)\rangle = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} dx_1\ldots dx_m\, f(x_1,\ldots,x_m)\prod_{j=1}^{m}\theta(y>x_j)u^{+}_{a_1}(x_1)\ldots u^{+}_{a_m}(x_m)\mid 0\rangle$$

is zero and hence (6.1.9) follows.

Appendix 3. The existence of $\Delta^{(a)}(x,\lambda)$ and $\Delta^{(a)}(\infty,\lambda)$

Noticing (A2.3), we find that the existence of $\Delta^{(a)}(x,\lambda)$ depends on the existence of

$$\chi^{+}_{m,N+1}(x,\lambda+ic(n-1)/2)\chi(n-1,r \mid b_1,\ldots,b_{n-r-1} \mid a_1,\ldots,a_{n-1} \mid \lambda-ic/2,x)$$
$$\equiv \chi(m \mid n-1,r \mid b_1,\ldots,b_{n-r-1} \mid a_1,\ldots,a_{n-1} \mid \lambda,x) \qquad \text{(A3.1)}$$

or $\chi(m \mid n-1,r \mid b_1,\ldots,b_{n-r-1} \mid \lambda,x)$ for short. With the help of

$$\chi(b \mid n,r-1 \mid b,b_1,\ldots,b_{n-r} \mid \lambda,x) = (-1)^r\chi(N+1 \mid n,r-1 \mid b_1,\ldots,b_{n-r} \mid \lambda,x) \qquad \text{(A3.2)}$$

which originates from

$$\chi^{+}_{b,N+1}(x,\lambda+ic)\chi_{a,b}(x,\lambda) = -\chi^{+}_{N+1,N+1}(x,\lambda+ic)\chi_{N+1,a}(x,\lambda) \qquad \text{(A3.3)}$$

we derive the following differential equations:

$$\frac{\partial}{\partial x}\chi(a \mid n,r \mid b_1,\ldots,b_{n-r} \mid \lambda,x)$$
$$= \tfrac{1}{2}i\left[(n-2r-1)\lambda-ic(n-r)(r+1)/2\right]\chi(a \mid n,r \mid b_1,\ldots,b_{n-r} \mid \lambda,x)$$
$$+\sqrt{-c}\,u^{+}_{a}(x)\chi(N+1 \mid n,r \mid b_1,\ldots,b_{n-r} \mid \lambda,x)$$
$$+\sqrt{-c}(-1)^{r-1}u^{+}_{b}(x)\chi(a \mid n,r-1 \mid a,b_1,\ldots,b_{n-r} \mid \lambda,x)$$
$$+\sqrt{-c}\sum_{l=1}^{n-r}(-1)^{l}\chi(a \mid n,r+1 \mid b_1,\ldots,\bar{b}_l,\ldots,b_{n-r} \mid \lambda,x)u_{b_l}(x) \qquad \text{(A3.4)}$$

$$\frac{\partial}{\partial x}\chi(N+1 \mid n, r \mid b_1, \ldots, b_{n-r} \mid \lambda, x)$$
$$= \tfrac{1}{2}\mathrm{i}\,[(n-2r+1)\lambda - \mathrm{i}c(n-r+1)r)]\,\chi(N+1 \mid n, r \mid b_1, \ldots, b_{n-r} \mid \lambda, x)$$
$$- \sqrt{-c}(-1)^r \chi(b \mid n, r \mid b_1, \ldots, b_{n-r} \mid \lambda, x) u_b(x)$$
$$+ \sqrt{-c}(-1)^r r u_b^+(x) \chi(N+1 \mid n, r-1 \mid b, b_1, \ldots, b_{n-r} \mid \lambda, x)$$
$$+ \sqrt{-c}\sum_{l=1}^{n-r}(-1)^l \chi(N+1 \mid n, r+1 \mid b_1, \ldots, \bar{b}_l, \ldots, b_{n-r} \mid \lambda, x) u_{b_l}(x). \qquad \text{(A3.5)}$$

By studying the Neumann expansion of $\chi(m \mid n, r \mid b_1, \ldots, b_{n-r-1} \mid \lambda, x)$ with help from (A3.4) and (A3.5), just as we did for $\varphi(\ldots)$, we can prove that $\chi(m \mid n, r \mid b_1, \ldots, b_{n-r-1} \mid \lambda, x)$ are analytic on the lower half λ plane. As a by-product, we prove the existence of $\Delta^{(a)}(\infty, \lambda)$.

References

Creamer D B, Thacker H B and Wilkinson D 1980 *Phys. Rev.* D **21** 1523
Faddeev L D and Sklyanin E K 1978 *Dokl. Akad. Nauk. SSSR* **243** 1430
Fan H, Pu F C and Zhao B H 1988 *Nucl. Phys.* B **299** 52
Gockeler M 1981a *Z. Phys.* C **7** 263
—— 1981b *Z. Phys.* C **11** 125
Izergin A G and Korepin V E 1981 *Sov. Phys.-Dokl.* **26** 653
Kulish P P 1980 *Dokl. Akad. Nauk. SSSR* **255** 323
—— 1985 *Lett. Math. Phys.* **10** 87
Pu F C and Zhao B H 1984 *Phys. Rev.* D **30** 2253
—— 1986 *Nucl. Phys.* B(FS) **275** (FS17) 77
Pu F C, Wu Y Z and Zhao B H 1987 *J. Phys. A: Math. Gen.* **20** 1173
Sklyanin E K 1979 *Dokl. Akad. Nauk. SSSR* **224** 1337
Thacker H B and D Wilkinson 1979 *Phys. Rev.* D **19** 3360
Zakharov V E and Shabat A B 1971 *Zh. Eksp. Teor. Fiz.* **61** 118 (1973 *Sov. Phys.-JETP* **34** 62)